C. Analysis M.Sc. Sem II
PAPER-VI

Cauchy's fundamental theorem

Th1. Cauchy's theorem

Let D be a simply connected region and let $f(z)$ be single valued continuously differential function on D i.e. $f'(z)$ exists and is continuous at each point of D. Then

$$\int_C f(z)\,dz = 0$$

where C is any closed contour contained in D.

Proof From Green's theorem for a plane

If $P(x,y)$, $Q(x,y)$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ are all continuous functions within a domain D and if C is any closed contour in D, then

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Now $\int_C f(z)\,dz = \int_C (u\,dx - v\,dy)$

$$+ i\int_C (v\,dx + u\,dy) \quad (1)$$

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We have $f'(z) = u_x + i v_x$

$= v_y - i u_y$ — (2)

By C-R Equation -

Since $f'(z)$ is continuously differentiable, it follows from (2) that u_x, u_y, v_x, v_y all exist and are continuous in D. Thus all the Conditions of Green's theorem are satisfied. Hence we get from (1)

$$\int_C f(z)\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$+ i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy$$

dxdy. $$+ i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$